PERGAMON

Topology 41 (2002) 321–340

TOPOLOGY

www.elsevier.com/locate/top

# All $\mathbb{Z}_q$ lens spaces have diffeomorphic squares

Slawomir Kwasik[a,*,1], Reinhard Schultz[b]

[a]Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

[b]Department of Mathematics, University of California, Riverside, CA 92521, USA

Received 15 July 1999; accepted 15 August 2000

## 0. Introduction

In 1933, Ulam (see [34]) posed the following problem:

**Problem.** *If $A$ and $B$ are topological spaces such that $A^2 = A \times A$ and $B^2 = B \times B$ are homeomorphic, are $A$ and $B$ homeomorphic?*

It is not difficult to see that in such generality this question has a negative answer. On the other hand, when one requires that the spaces involved are more regular (i.e., compact connected manifolds or polyhedra) then the above problem becomes much more challenging. But even in this case the answer is negative. In 1948, Fox (see [9]) constructed two four-dimensional compact, connected manifolds which are nonhomeomorphic but whose Cartesian squares are homeomorphic. After that, other such examples were constructed by Glimm [11], Kwun [18], and McMillan [20].

It turns out, however, that Ulam's problem has a positive solution for compact 2-manifolds (cf. [5,9]). A much deeper result of Rosicki [26] gives a positive answer to Ulam's question for compact connected two-dimensional polyhedra. His paper also contains the following:

**Question.** *Do there exist nonhomeomorphic, compact, connected three-dimensional polyhedra $A$ and $B$ such that $A \times A$ and $B \times B$ are homeomorphic?*

A result of Sieradski [31] shows that if $L$ and $L'$ are two three-dimensional lens spaces with isomorphic fundamental groups then $L \times L$ and $L' \times L'$ are simple homotopy equivalent. To put

[1] Partially supported by NSF Grants INT-9505890 and INT-9808925. Version of June 14, 2000.

* Corresponding author. Tel.: + 1-504-862-3437; fax: + 1-504-865-5063.
*E-mail address:* kwasik@math.tulane.edu (S. Kwasik).



PII: S0040-9383(00)00039-2

this in perspective, recall that homotopy equivalent lens spaces need not be simply homotopy equivalent (see [8]). In this paper, we shall answer Rosicki's question affirmatively by showing that the squares are in fact diffeomorphic.

**Theorem A.** *Let $q > 1$ be given, let $a$, $b$ be integers prime to $q$, and let $L(q;\, a)$ and $L(q;\, b)$ be the lens spaces associated to the representations of $\mathbb{Z}_q$ defined by*

$$c\cdot(z_1, z_2) = (cz_1, c^a z_2),$$

$$c\cdot(z_1, z_2) = (cz_1, c^b z_2),$$

*respectively. Then $L(q;\, a)\times L(q;\, a)$ and $L(q;\, b)\times L(q;\, b)$ are diffeomorphic.*

Since two finite cyclic groups have isomorphic squares if and only if the groups themselves are isomorphic, it follows that squares of three-dimensional lens spaces are classified up to diffeomorphism by their fundamental groups.

In fact, our methods lead to a complete classification of $n$th powers of lens spaces, where $n$ is an arbitrary integer $\geqslant 2$. Given a space or smooth manifold $A$ we shall denote its $n$th Cartesian power ( = its $n$-fold Cartesian product with itself) by $\prod^n A$.

**Theorem B.** *Let $L$ and $L'$ be two three-dimensional lens spaces, and let $n \geqslant 2$ be an integer. If $n$ is even, then $\prod^n L$ is diffeomorphic (resp. homeomorphic) to $\prod^n L'$ if and only if the fundamental groups of $L$ and $L'$ are isomorphic. If $n$ is odd, then $\prod^n L$ is diffeomorphic (resp. homeomorphic) to $\prod^n L'$ if and only if $L$ and $L'$ are homotopy equivalent.*

The principal ingredients in the proofs of the main results are surgery theory and an analysis of the normal invariants of certain homotopy equivalences of squares of lens spaces. Both the homotopy-theoretic and surgery-theoretic considerations turn out to require some fairly detailed results about the behavior of surgery obstructions and normal invariants. If the fundamental group of the lens spaces has odd order the proof is relatively simple, and we begin by disposing of this case in Section 1. In Section 2, an analysis of surgery exact sequences and computational information about their behavior under certain transfer homomorphisms are used to show that certain simple homotopy equivalences of products of two lens spaces are homotopic to diffeomorphisms if their normal invariants are trivial; not all of these products are squares. The results of Section 3 imply that certain homotopy equivalences from $L(q;\, a)\times L(q;\, a)$ to $L(q;\, b)\times L(q;\, b)$ considered by Sieradski [31] have trivial normal invariants if $q$ is divisible by 8 and $ab \equiv \pm 1 \bmod 8$, while if $q$ is divisible by 8 and $ab \equiv \pm 3 \bmod 8$ then the simple homotopy equivalences of [31] have nontrivial normal invariants. In order to complete the proof, it is necessary to compose certain Sieradski homotopy equivalences with simple homotopy self-equivalences having the same normal invariants. A relatively elementary construction of such self-equivalences is given in Section 5, where the proof of Theorem A is completed. In Section 6, we prove Theorem B and show that the results on lens spaces lead to a negative answer to Ulam's question in every dimension $\geqslant 4$. The methods of [9] strongly suggest the existence of examples in all higher dimensions but this fact does not seem to be established in the literature.

The lens spaces considered here are examples of Seifert fibered 3-manifolds. This class of manifolds is large and important in the theory of 3-manifolds, and it is natural to ask if there are other examples of nonhomeomorphic closed 3-manifolds — and particularly Seifert 3-manifolds — with diffeomorphic or homeomorphic squares. In [17], we show that no other examples exist within the class of Seifert 3-manifolds, and we also show this for another family of 3-manifolds that includes all lens spaces; namely, the class of all irreducible geometric 3-manifolds that are rational homology spheres or connected sums of such manifolds that have no lens spaces in their prime decompositions.

The first examples of nonstandard diffeomorphisms between products of three-dimensional lens spaces were constructed by Metzler [21,22], and some of his methods were used by Sieradski in his work. One major difference between Metzler's results and ours is that his generally involve products where the fundamental groups of the factors have relatively prime orders.

## 1. Odd order fundamental groups

Let $L$ and $L'$ be lens spaces with isomorphic fundamental groups and suppose that $f: L\times L \to L'\times L'$ is a simple homotopy equivalence. For each prime divisor $p$ of the order $q$ of $\pi_1(L) \approx \pi_1(L')$, let $L\langle p\rangle$ and $L'\langle p\rangle$ denote the covering spaces associated to the Sylow $p$-subgroup of the fundamental group. Since $L\times L$ and $L'\times L'$ are even dimensional and have finite abelian fundamental groups, by Kwasik and Schultz [16, Theorem B($i$)], the map $f$ is homotopic to a diffeomorphism if for each prime divisor $p$, the liftings

$$f\langle p\rangle : L\langle p\rangle \times L\langle p\rangle \to L'\langle p\rangle \times L'\langle p\rangle$$

of $f$ to the coverings associated to the Sylow $p$-subgroups are homotopic to diffeomorphisms.

Surgery theory is generally much easier to apply for fundamental groups of odd order than for arbitrary finite groups, and therefore we shall begin by proving the main result for lens spaces whose fundamental groups have odd order.

**Theorem 1.1.** *Let $q$ be an odd integer $\geqslant 3$, and $L$ and $L'$ be two lens spaces whose fundamental groups have order $q$. Then $L\times L$ is diffeomorphic to $L'\times L'$.*

**Proof.** By the results of Sieradski we know that $L\times L$ and $L'\times L'$ are simple homotopy equivalent [31, Theorem 3, p. 86]. Let $h: L\times L\to L'\times L'$ be a simple homotopy equivalence. We want to show that we can choose $h$ to be homotopic to a diffeomorphism. For the time being assume that $h$ is an arbitrary simple homotopy equivalence. Consider the long exact surgery sequence:

$$\cdots \to L_7^s(\mathbb{Z}_q\times\mathbb{Z}_q) \xrightarrow{\gamma} S_{DIFF}(L'\times L') \xrightarrow{\eta} [L'\times L', G/O] \xrightarrow{\theta} L_6^s(\mathbb{Z}_q\times\mathbb{Z}_q).$$

Since $L_7^s(\mathbb{Z}_q\times\mathbb{Z}_q)=0$ (cf. [1]), it suffices to show that $\eta([h])=0$ in $[L'\times L', G/\mathrm{Top}]$.

A simple computation gives

$$[L' \times L', G/O] \approx H^4(L' \times L'; \mathbb{Z}) \oplus H^6(L' \times L'; \mathbb{Z}_2) \approx \mathbb{Z}_q \oplus \mathbb{Z}_2,$$

where the additive structure is given by the direct sum $H$-space structure on $G/O$. Moreover, the copy of $\mathbb{Z}_2$ goes nontrivially to $L_6^s(\mathbb{Z}_q \times \mathbb{Z}_q)$ under $\theta$, because it is detected by the classical Arf invariant.

Since both $L'$ and $L$ are parallelizable, it follows that $\eta([h])$ in fact lies in the image of $[L' \times L'; G]$. Therefore, it would suffice to show that the map

$$[L' \times L', G/O] \to [L' \times L'; BO]$$

given by taking the underlying (stable) vector bundle is trivial on elements of odd order.

We shall prove that the latter holds if $q$ is not divisible by 3 and that it nearly holds if $q$ is divisible by 3. Recall (cf. [4]) that the map $G/O \to BO$ given by taking the underlying vector bundle is a map of infinite loop spaces. This is helpful for us because we can now use the splitting of $L'$ as a wedge $X \vee S^3$ in the category of finite spectra to write $[L' \times L', G/O]$ and $[L' \times L', BO]$ as direct sums; the space $X$ is just the Moore space $S^1 \cup_q e^2$. With respect to this splitting $H^4(L' \times L'; \mathbb{Z})$ comes from a stable summand of $L' \times L'$ of the form $X \wedge X$, and thus by naturality it suffices to consider the map

$$[X \wedge X, G/O]_{\{\mathrm{odd}\}} \to [X \wedge X; BO]_{\{\mathrm{odd}\}}.$$

The domain and codomain of this map are both isomorphic to $\mathbb{Z}_q$, and in fact the homomorphism is given by tensoring the map

$$\mathbb{Z} \approx \pi_4(G/O) \to \pi_4(BO) \approx \mathbb{Z}$$

with $\mathbb{Z}_q$. Since the map of homotopy groups is multiplication by 24, it follows that the kernel of the map of interest to us has exponent dividing 24. If 3 does not divide the odd integer $q$, it follows that the kernel is trivial, and accordingly the conclusion of Theorem 1.1 follows in these cases.

Suppose now that 3 does divide $q$. Let $r = 3q$, and let $M$ and $M'$ be lens spaces such that their fundamental groups have order $r$ and $M$ and $M'$ have 3-sheeted coverings given by $L$ and $L'$, respectively. Furthermore, let $Y$ be the stable Moore space summand $S^1 \cup_r e^2$ of $M'$ analogous to $X$. If $H$ is an arbitrary simple homotopy equivalence from $M \times M$ to $M' \times M'$, then $H$ lifts to a simple homotopy equivalence $H^*$ from $L \times L$ to $L' \times L'$, and the normal invariant of $H^*$ is just the pullback of the normal invariant of $H$ under the covering projection from $M \times M'$ to $L \times L'$. Cellular approximation implies that the covering space projection from $M'$ to $L'$ may be assumed to send $X$ into $Y$, and therefore we have a commutative diagram of the following form:

$$\begin{array}{ccc} [X \times X; G/O]_{\{\mathrm{odd}\}} & \longrightarrow & [X \times X, BO]_{\{\mathrm{odd}\}} \\ \uparrow & & \uparrow \\ [Y \times Y; G/O]_{\{\mathrm{odd}\}} & \longrightarrow & [Y \times Y, BO]_{\{\mathrm{odd}\}} \end{array}$$

In this diagram the horizontal arrows correspond to multiplication by 24 and the vertical arrows are merely the usual surjections from $\mathbb{Z}_r$ to $\mathbb{Z}_q$. By the preceding discussion the normal invariant of $H$ is an element whose order divides 3 in $[Y \times Y, G/O]_{\{\mathrm{odd}\}} \approx \mathbb{Z}_r$, and we also know that the image of this element in $[X \times X, G/O]_{\{\mathrm{odd}\}} \approx \mathbb{Z}_q$ is the normal invariant of $H^*$. Therefore, it follows that the normal invariant of the homotopy equivalence $H^*$ must vanish when $q$ is divisible by 3, and therefore the conclusion of Theorem 1.1 also follows. □

## 2. Surgery obstructions

In the proof of the main theorem for odd order fundamental groups, two important simplifications are that the surgery obstruction groups have relatively simple descriptions and the normal invariants are easy to handle. An extension of the main result to even order fundamental groups will require a more detailed analysis of the roles of these two aspects of the surgery exact sequences for squares of lens spaces. This section is devoted to studying the Wall groups $L_3^s(\mathbb{Z}_q \times \mathbb{Z}_q, 1)$ and their actions in the surgery exact sequences; in Section 3 there will be a corresponding analysis of the normal invariants.

The analysis of simple homotopy equivalences of lens space squares requires a notion of diagonalization for the associated isomorphisms of fundamental groups. To develop the latter, we need the concept of a *polarization* of an arcwise connected space $X$ with fundamental group $\pi$; this is technically a homotopy class of maps from $X$ to $K(\pi,1)$ that induces an isomorphism of fundamental groups, but by abuse of language we shall refer to a map itself as a polarization.

There is a standard polarization of $L(q;a) = S^3/\mathbb{Z}_q$ obtained from the isometric $\mathbb{Z}_q$ action on $S^3$ that defines the lens space, and we shall denote it by $k_a$. Of course, the product of two polarizations $X \to K(\pi,1)$ and $X' \to K(\pi',1)$ is again a polarization, and this yields canonical polarizations of lens space squares.

**Theorem 2.1.** *Let $q \geqslant 2$ be an integer divisible by 8, let $a$, $a'$ and $b$ be integers that are prime to $q$, and let $f\colon L(q;a) \times L(q;a') \to L(q;b) \times L(q;b)$ be a simple homotopy equivalence satisfying the following conditions*:

(i) *$f$ is normally cobordant to the identity.*

(ii) *There is a homotopy commutative diagram $[\mathbf{C}_q]$ of the form*

$$\begin{array}{ccc} L(q;a) \times L(q;a') & \xrightarrow{f} & L(q;b) \times L(q;b) \\ \downarrow{\scriptstyle k(a,a')} & & \downarrow{\scriptstyle k(b,b)} \\ K(\mathbb{Z}_q,1) \times K(\mathbb{Z}_q,1) & \xrightarrow{g \times g'} & K(\mathbb{Z}_q,1) \times K(\mathbb{Z}_q,1) \end{array}$$

*where $k(u,v)$ denotes the canonical polarization and $g, g'$ are homotopy self equivalences of $K(\mathbb{Z}_q,\ 1)$.*
(iii) *$f$ covers a simple homotopy equivalence*

$$\bar{f}: L(2q;\ a) \times L(2q;\ a') \to L(2q;\ b) \times L(2q;\ b)$$

*with the same properties, and the covering space projections define a map from diagram $[\mathbf{C}_q]$ to diagram $[\mathbf{C}_{q+1}]$. THEN $f$ is homotopic to a diffeomorphism.*

One can choose the simple homotopy equivalences of [31] so that the hypotheses of the proposition hold, so the result will apply to some (in fact, infinitely many) of these maps.

**Proof.** Write $q = 2^k r$ (so that $k \geqslant 3$), and let $M_k = L(2^k r;\ a)$, $N_k = L(2^k r;\ a')$, $L_k = L(2^k r;\ b)$, $M_{k+1} = L(2^{k+1}r, a)$, $N_{k+1} = L(2^{k+1}r, a')$ and $L_{k+1} = L(2^{k+1}r, b)$. The covering hypothesis implies there is a commutative diagram

$$\begin{array}{ccc} M_k \times N_k & \xrightarrow{\quad f \quad} & L_k \times L_k \\ \big\downarrow{\scriptstyle \pi_M \times \pi_N} & & \big\downarrow{\scriptstyle \pi_L \times \pi_L} \\ M_{k+1} \times N_{k+1} & \xrightarrow{\quad \bar{f} \quad} & L_{k+1} \times L_{k+1} \end{array}$$

where $\pi_M$, $\pi_N$ and $\pi_L$ are the 2-fold coverings. We claim that $f$ is homotopic to a diffeomorphism. Consider the commutative diagram of surgery exact sequences.

$$\begin{array}{ccccc} \ldots & \longrightarrow L_3^s(Z_{2^k r} \times Z_{2^k r}) & \xrightarrow{\gamma_k} & S_{\mathrm{Top}}(L_k \times L_k) & \xrightarrow{\eta_k} \\ & \big\uparrow{\scriptstyle tr} & & \big\uparrow{\scriptstyle tr} & \\ \ldots & \longrightarrow L_3^s(Z_{2^{k+1} r} \times Z_{2^{k+1} r}) & \xrightarrow{\gamma_{k+1}} & S_{\mathrm{Top}}(L_{k+1} \times L_{k+1}) & \xrightarrow{\eta_{k+1}} \end{array}$$

$$\begin{array}{cccc} \xrightarrow{\eta_k} & [L_k \times L_k; G/\mathrm{Top}] & \xrightarrow{\theta_2} & L_2^s(Z_{2^k r} \times Z_{2^k r}) \\ & \big\uparrow{\scriptstyle tr} & & \big\uparrow{\scriptstyle tr} \\ \xrightarrow{\eta_{k+1}} & [L_{k+1} \times L_{k+1}; G/\mathrm{Top}] & \longrightarrow & L_2^s(Z_{2^{k+1} r} \times Z_{2^{k+1} r}) \ . \end{array}$$

Clearly, $[M_k \times N_k, f\,] = tr[M_{k+1} \times N_{k+1}, \bar{f}\,]$, and by assumption we know that $\eta_k[M_k \times N_k, f\,]$ and $\eta_{k+1}[M_{k+1} \times N_{k+1}, \bar{f}\,]$ both vanish. Therefore, it remains to show that $[M_k \times N_k, f\,]$ cannot be written as a nontrivial class of the form $\gamma_k(\alpha)$ for some $\alpha \in L_3^s(Z_{2^k r} \times Z_{2^k r})$.

Write $\mathbb{Z}_q$ as $\mathbb{Z}_{2^k} \times \mathbb{Z}_r$. Then $\mathbb{Z}_q \times \mathbb{Z}_q \approx \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k} \times \mathbb{Z}_r \times \mathbb{Z}_r$. The result in [36, Theorem 2.4.2], implies $L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k} \times \mathbb{Z}_r \times \mathbb{Z}_r) \approx L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$. This and a transfer argument when applied to $[L_k \times L_k, G/\mathrm{TOP}]$ imply that, without loss of generality, we can assume $\pi_1(L_k) \approx \mathbb{Z}_{2^k}$ for some $k \geqslant 1$. Now, Theorem 3.3.2 in [36] or Theorem 6 in [2] gives $L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In order to prove Theorem 2.1, we must show that $[M_k \times N_k, f\,] \notin im\, \gamma_k - \{[L_k \times L_k, id]\}$, where $\gamma_k: L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \to S_{\mathrm{Top}}(N \times N)$ is the usual Wall group action.

Our first observation is that $im\, \gamma_k \subseteq \mathbb{Z}_2$. Namely, it turns out that two copies of $\mathbb{Z}_2$ in $L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$ are in the so-called “Ooze” subgroup. In other words, these two copies of $\mathbb{Z}_2$ are in the images of the homomorphisms $\theta_2:[\Sigma(L_k \times L_k), G/\mathrm{TOP}] \to L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$ and its smooth analog $\theta_2^{\mathrm{Diff}}:[\Sigma(L_k \times L_k), G/O] \to L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$ in the long exact surgery sequences. There probably are several ways of seeing this; one possibility is as follows:

If $W$ is any of the lens spaces under consideration, then a simple computation (cf. [30, p. 155]) gives

$$[\Sigma(W \times W);G/\mathrm{TOP}] \approx H^6(\Sigma(W \times W);\mathbb{Z}_2) \oplus H^4(\Sigma(W \times W);\mathbb{Z}\ ) \oplus H^2(\Sigma(W \times W);\mathbb{Z}_2).$$

In particular, $H^6(\Sigma(W \times W);\mathbb{Z}_2) \approx H^5(W \times W;\mathbb{Z}_2) \approx H_1(W \times W;\mathbb{Z}_2) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $f_i:X_i \to \Sigma(W \times W)$, $i = 1,2,3,4$, be normal maps with normal invariants classified by $H_1(W \times W;\ \mathbb{Z}_2)$. In the notation of ([12, p. 351]), these maps are classified by their *Arf* invariants $ARF_1(f_i) \in H_1(W \times W;\mathbb{Z}_2)$. Now, Theorem A(d) in ([12, p. 351]) asserts that the surgery obstruction $\theta(f_i) \in L_3^h(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$ of each of these maps is essentially determined by $\kappa_1^h(ARF_1(f_i))$, where

$$\kappa_1^h:H_1(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k};\mathbb{Z}_2) \to L_3^h(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$$

is an universal homomorphism constructed in [12]. (The homomorphism $c_*$ in Theorem A(d) can be disregarded in our case.) Since the homomorphism $\kappa_1^h$ is a monomorphism (cf. [12, Theorem 6.8]), then indeed the images of the smooth and topological surgery obstructions into $L_3^h(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$ are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The results are the same for smooth and topological surgery because the map $\pi_6(G/O) \to \pi_6(G/\mathrm{TOP})$ is an isomorphism.

Now the commutativity of the diagram

$$\begin{tikzcd}
\theta_k^s : [\Sigma(W \times W); G/\mathrm{Top}] \arrow[r] \arrow[d, equal] & L_3^h(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \\
\theta_k^h : [\Sigma(W \times W); G/\mathrm{Top}] \arrow[r] & L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \arrow[u, "F"']
\end{tikzcd}$$

(where $F$ is the forgetful homomorphism in the Rothenberg exact sequence) implies that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset Ooze\ (L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}))$, and hence $im\,\gamma_k$ is either trivial or isomorphic to $\mathbb{Z}_2$. If $im\,\gamma_k = 0$, then our claim follows; therefore, assume $im\,\gamma_k \approx \mathbb{Z}_2$.

We shall show now that one copy of $\mathbb{Z}_2 \subset Ooze\ L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$, $k \geqslant 2$, is determined by a copy of $\mathbb{Z}_2$ in $L_3^s(\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_{2^k}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, under the transfer

$$tr_*:L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \to L_3^s(\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_{2^k})$$

which is induced by the standard inclusion

$$\mathbb{Z}_{2^{k-1}} \hookrightarrow \mathbb{Z}_{2^k}.$$

To prove this claim, we appeal to the following "iterated transfer" argument. Express

$$tr_*:L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \to L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k})$$

as a composition of transfers

$$L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \to L_3^s(\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_{2^k}) \to \cdots \to L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k})$$

each being induced by the inclusion of the index two subgroup of the form

$$\mathbb{Z}_{2^{i-1}} \times \mathbb{Z}_{2^k} \hookrightarrow \mathbb{Z}_{2^i} \times \mathbb{Z}_{2^k}.$$

It follows from Theorem 5 in [23] that the image of the transfer homomorphism on each stage

$$L_3^s(\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^k}) \to L_3^s(\mathbb{Z}_{2^{i-1}} \times \mathbb{Z}_{2^k})$$

consists of a copy of $\mathbb{Z}_2$. Moreover, this copy of $\mathbb{Z}_2$ survives under the composition of consecutive transfers. To see this, let $t$ be a generator of $\mathbb{Z}_{2^i}$ and let $\tau = t^2$ be a generator of $\mathbb{Z}_{2^{i-1}}$.

The Tate cohomology groups

$$H^0(\hat{\mathbb{Z}}_2[\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^k}]), \quad H^0(\hat{\mathbb{Z}}_2[\mathbb{Z}_{2^{i-1}} \times \mathbb{Z}_{2^k}])$$

determine elements in the corresponding $L$-groups $L_3^s(\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^k})$, $L_3^s(\mathbb{Z}_{2^{i-1}} \times \mathbb{Z}_{2^k})$ via the version of a Rothenberg exact sequence (cf. [23]). In particular, the elements $(1 + t + t^{-1})$ and $(1 + \tau + \tau^{-1})$ represent generators in $\mathbb{Z}_2 \subset L_3^s(\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^k})$ and $\mathbb{Z}_2 \subset L_3^s(\mathbb{Z}_{2^{i-1}} \times \mathbb{Z}_{2^k})$, respectively (Lemma 5, p. 345). Moreover,

$$tr_*(1 + t + t^{-1}) = (1 + \tau + \tau^{-1}) \qquad \text{(Lemma 5 once again).}$$

Consider now the commutative diagram:

$$\begin{array}{ccc} \theta : [\Sigma(W \times W); G/\mathrm{Top}] & \longrightarrow & L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \\ \big\downarrow{\scriptstyle tr_*} & & \big\downarrow{\scriptstyle tr_*} \\ \theta : [\Sigma(\mathbb{R}P^3 \times L_k); G/\mathrm{Top}] & \longrightarrow & L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) \end{array}$$

We want to show that the composite in this diagram is onto $L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By symmetry it suffices to find preimages for each $\mathbb{Z}_2$ summand. Since $tr_*|:H_1(L_k \times L_k;\, \mathbb{Z}_2) \to H_1(\mathbb{R}P^3 \times L_k;\mathbb{Z}_2)$ is nontrivial, and in fact

$$im\, tr_* \approx \mathbb{Z}_2 \approx tr_*((H_1(L_k;\mathbb{Z}_2) \otimes H_0(L_k;\mathbb{Z}_2)) \approx H_1(\mathbb{R}P^3,\mathbb{Z}_2) \otimes H_0(L_k;\, \mathbb{Z}_2) \approx \mathbb{Z}_2$$

it follows that the image of the composite is equal to $Ooze\, L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k})$. Note that $Ooze\, L_3^s(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (i.e., Theorem A in [12] applies here because $SK_1(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) \approx 0$ by Oliver [25, Theorem 14.2, p. 330]) and that the *Ooze* subgroup is transfer invariant. By symmetry it follows that $im\, tr_* \approx \mathbb{Z}_2$ for

$$tr_*: L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \to L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_2)$$

and in particular, for $\mathbb{Z}_2 \approx H \not\subseteq Ooze\, L_3^s(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})$,

$$tr_*(H) = 0.$$

To show that $[M_k \times N_k, f] \notin im\,\gamma_k - \{[L_k \times L_k, id]\}$, observe that conditions (ii) and (iii) in the statement of the proposition yield the following commutative diagram:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
[\Sigma(L_k \times L_{k+1}); G/O] & \xrightarrow{tr_*} & [\Sigma(L_k \times L_k); G/O] \\
\downarrow \theta^1_{k+1} & & \downarrow \theta^1_k \\
L^s_3(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{k+1}}) & \xrightarrow{tr_*} & L^s_3(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}) \\
\downarrow \gamma_{k+1} & & \downarrow \gamma_k \\
S_{\mathrm{Diff}}(L_k \times L_{k+1}) & \xrightarrow{tr_*} & S_{\mathrm{Diff}}(L_k \times L_k) \\
\downarrow \eta_{k+1} & & \downarrow \eta_k \\
[L_k \times L_{k+1}; G/O] & \xrightarrow{tr_*} & [L_k \times L_k; G/O] \\
\downarrow \theta_{k+1} & & \downarrow \theta_k \\
L^s_2(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{k+1}}) & \xrightarrow{tr_*} & L^s_2(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k})
\end{array}$$

By conditions (ii) and (iii) the map $\bar{f}$ lifts to a simple homotopy equivalence

$$f^{\#}: M_k \times N_{k+1} \to L_k \times L_{k+1}$$

such that the lifting of the latter to $M_k \times N_k$ is $f$, and clearly $[M_k \times N_{k+1}, f^{\#}] \in S_{\mathrm{Top}}(L_k \times L_{k+1})$ and

$$tr_*([M_k \times N_{k+1}, f^{\#}]) = [M_k \times N_k, f] \in S_{\mathrm{Diff}}(L_k \times L_k).$$

As before, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset Ooze\, L^s_3(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{k+1}})$ and we can assume $[M_k \times L_{k+1}, f^{\#}] \in im\,\gamma_{k+1} \approx \mathbb{Z}_2$. If $\mathbb{Z}_2 \approx H \nsubseteq Ooze\, L^s_3(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{k+1}})$, then

$$tr_* \gamma_{k+1}(H) = tr_*([M_k \times N_{k+1}, f^{\#}]) = [M_k \times N_k, f] = \gamma_k\, tr_*(H) = \gamma_k(0) = 0.$$

This implies $[M_k \times N_k, f]$ is trivial in $S_{\mathrm{Diff}}(L_k \times L_k)$, so in particular $f$ is homotopic to a diffeomorphism. $\square$

## 3. Normal invariants of Sieradski's maps

The results of Section 2 reduce the proof of the main result to studying the normal invariants of the simple homotopy equivalences of lens space squares constructed by Sieradski in [31]. Since lens spaces are orientable 3-manifolds and all such manifolds are parallelizable, it follows that all homotopy equivalences between squares of lens spaces are tangential (i.e., the pullback of the tangent bundle of the target manifold is the tangent bundle of the source). It is well known [6] that closed parallelizable manifolds are equivalent to their own Spanier–Whitehead duals in stable homotopy theory, and in fact it is also well known (see [19]) that the normal invariant of a homotopy equivalence of closed parallelizable manifolds is given by

$$\beta_* \alpha(D(h \vee S^0)), \tag{3.1}$$

where $D(h \vee S^0)$:$N \vee S^0 \to M \vee S^0$ is the Spanier–Whitehead dual of $h \vee S^0$, the map $\alpha$:$\{N \vee S^0, M \vee S^0\} \to \{N, S^0\}$ sends a class $H$:$M \vee S^0 \to N \vee S^0$ to the projection of $H \,|\, N$ onto $S^0$, and

$$\beta_*\colon \{N, S^0\} \approx [N, G] \to [N, G/O]$$

is induced by the composite of the map $\Omega^\infty S^\infty \to G$ essentially given by track sum with the identity map and the canonical projection $G \to G/O$ (for example, this is implicit in [24, p. 315]; [27, definition of $r_1$, p.140]). Even though the domain and codomain of $\beta_*$ have natural abelian group structures, the map $\beta_*$ is not necessarily additive. By construction, the composite of the standard inclusion $[N, SO] \to [N, G]$ with the inverse of the isomorphism $\{N, S^0\} \approx [N, G]$ is given by the classical $J$-homomorphism.

Given a positive integer $r$ prime to $q$ and a second integer $d$ congruent to $r \bmod q$, by obstruction theory there is a unique homotopy class of maps $u$ from $L(q;\ a)$ to $L(q;\ ad)$ that is polarization preserving (in the sense of Section 2; i.e., $k_{ad} \circ u = k_a$) and has degree $d$. An explicit representative is given by the map $\psi^d$:$L(q;\ a) \to L(q;\ ad)$ that lifts to the map of universal covering spaces $\tilde{\psi}^d$:$S^3 \to S^3$ given by

$$\tilde{\psi}^d(z_1, z_2) = (z_1^d, z_2).$$

These maps play a crucial role in Sieradski's construction of simple homotopy equivalences from $L(q;\ a) \times L(q;\ a)$ to $L(q;\ 1) \times L(q,\ 1)$. His maps admit canonical factorizations of the form

$$L(q;\ a) \times L(q;\ a) \xrightarrow{G} L(q;\ a^2) \times L(q,\ 1) \xrightarrow{f \times 1} L(q;\ 1) \times L(q;\ 1),$$

where $f$:$L(q;\ a^2) \to L(q;\ 1)$ is a homotopy equivalence and $G$ is a polarization preserving simple homotopy equivalence describable as follows: Let $P$:$L(q;\ a) \to L(q;\ a) \vee S^3$ be the pinching map defined by collapsing the boundary of a smoothly embedded closed 3-disk in $L(q;\ a)$ to a point; then the codomain of the squared map $P \times P$ is a union of the four closed subspaces:

$$L(q;\ a) \times L(q;\ a),$$

$$S^3 \times L(q;\ a),$$

$$L(q;\ a) \times S^3,$$

$$S^3 \times S^3$$

and the intersection of each pair of subspaces lies in $L(q;\ a) \vee L(q;\ a)$. Using this decomposition one defines a map $\Phi$ from $(L(q;\ a) \vee S^3) \times (L(q;\ a) \vee S^3)$ to $L(q;\ a^2) \times L(q;\ 1)$ by starting with $\psi^a \times \psi^t$ on $L(q;\ a) \times L(q;\ a)$. By construction, $ta = kq^2 + 1$ for some integer $k$, and we define $\Phi$ on $S^3 \times L(q;\ a)$ by the composite

$$S^3 \times L(q;\ a) \xrightarrow{(\deg k) \vee \psi^t} S^3 \times L(q;\ ta) \xrightarrow{\mu} \{pt.\} \times L(q;\ 1) \subseteq L(q;\ a^2) \times L(q;\ 1), \tag{3.2}$$

where $\mu$ is the usual transitive action of $S^3$ on the simple lens space $L(q;\ a)$. Likewise, the definition of $\Phi$ on $L(q;\ a)\times S^3$ is given by a composite

$$S^3\times L(q;\ a)\xrightarrow{1\times\psi^a} S^3\times L(q;\ a^2)\xrightarrow{\mu'} L(q;\ a^2)\times\{pt.\}\subseteq L(q;\ a^2)\times L(q;\ 1), \tag{3.3}$$

where $\mu'$ is an extension of the map $S^3\vee L(q;\ a^2)\to L(q;\ a^2)$ given by the universal covering on the first summand and the identity on the second; the existence of $\mu'$ is established in [31, Proposition 2, p. 90]. Finally, on $S^3\times S^3$ the map is given by

$$S^3\times S^3\xrightarrow{\text{twist}} S^3\times S^3\xrightarrow{\pi\times k\pi'} L(q;\ a^2)\times L(q;\ 1), \tag{3.4}$$

where $\pi$ and $\pi'$ denote the universal coverings, and $k\pi'$ is the $k$-fold sum of $\pi'$ in $\pi_3(L(q;\ 1))$. It follows that the composite $G=\Phi\circ(P\times P)$ is a polarization preserving map and that the matrix of

$$G_*/\text{Torsion}{:}H_3(L(q;\ a)\times L(q;\ a))/\text{Torsion}\to H_3(L(q;\ a^2)\times L(q;\ a))/\text{Torsion}$$

with respect to the standard Künneth Formula basis has the form

$$\begin{pmatrix} q & kq\\ q & t\end{pmatrix}.$$

Since this matrix has determinant 1, it follows that $G$ is a homotopy equivalence. For purposes of computing the normal invariant of $G$, the following result is helpful:

**Proposition 3.5.** *The homotopy equivalence $G$ is normally cobordant to the disjoint union of $\psi^a\times\psi^t$ with* (3.2)–(3.4).

**Sketch of Proof.** Given any closed manifold $M^n$, the boundary connected sum $N=M\times I\ \#\ S^n\times I$ defines a cobordism from $M^n$ to $M^n\coprod S^n$ and has the homotopy of $M\vee S^n$, and if $M$ is parallelizable, then so is the cobordism. Using this, one can construct a cobordism

$$W=M\times N\cup_{M\times\partial_+N}(N\times(M\textstyle\coprod S^n))$$

(parallelizable, if appropriate) from $M\times M$ to

$$(M\times M)\textstyle\coprod(S^n\times M)\coprod(M\times S^n)\coprod(S^n\times S^n)$$

with a canonical map to $(M\vee S^n)\times(M\vee S^n)$. Specializing to $M=L(q;\ a)$, we obtain a cobordism homotopy equivalent to the domain of $\Phi$, and it is a routine exercise to verify that the restrictions of the corresponding map $\Phi'$ (on $W$) to the two ends are $G$ and the map in the statement of the proposition. $\square$

Given a map $f:N\to M$ of closed parallelizable manifolds of the same dimension, the refined normal invariant $\tilde\eta(f)$ of $f$ will be the image of the composite $\alpha D(f\vee S^0)$ in $\{M,S^0\}$ described previously. One useful feature of this concept is that $\tilde\eta$ sends maps on disjoint unions to sums in stable cohomotopy. This and Proposition 3.5 yield a strategy for computing $\tilde\eta(G)$ in terms of the refined normal invariants of the pieces. The most important piece involves the map $\psi^a \times \psi^t$. Since the refined normal invariant construction sends cartesian products into cup products in stable cohomology, it suffices to compute the effect of the refined normal invariant construction on a map $\psi^r: L(q;\ a) \to L(q;\ ra)$. Since the Sullivan class map $k_2: G/O \to K(\mathbb{Z}_2, 2)$ is 4-connected, it follows that the normal invariant is determined by its $k_2$ class in $H^2(L(q;\ ra); \mathbb{Z}_2)$, which is 0 if $q$ is odd and $\mathbb{Z}_2$ if $q$ is even. The following result is an immediate consequence of the methods of [29]:

**Proposition 3.6.** *Let $q$ be even, let $a$ and $r$ be prime to $q$, and let $\psi^r$ be as before. Then $k_2\eta(\psi^r)$ is zero if $r \equiv \pm 1 \bmod 8$ and $[L(q;\ ra), G/O] \approx H^2(L(q;\ ra); \mathbb{Z}_2) \approx \mathbb{Z}_2$.*

Furthermore, the methods of [28, Section 1] imply that this localized normal invariant is the image of the previously discussed refined normal invariant under the composite

$$\{L, S^0\} \to [L, G/O] \xrightarrow{\cong} [L, G/O]_{(2)}$$

(division by $r$ in [28, Section 1] changes nothing because the normal invariants lie in $\mathbb{Z}_2$).

**Proposition 3.7.** *Let $q, a, r$ be as above, and let $\psi^r$ be as before. Then $k_2\tilde{\eta}(\psi^r)$ is zero if $r \equiv \pm 1 \bmod 8$ and nonzero if $r \equiv \pm 3 \bmod 8$.*

**Proof.** By construction the map $\psi^r$ is an $r$-fold branched covering of the sort considered in [29, Section 3], and therefore the results and methods of [29, especially Theorem 3.3, p. 597] imply that the 2-local normal invariant of $\psi^r$ is given by the composite

$$L(q;\ ra) \xrightarrow{\subseteq} K(Z_q,\ 1) \subseteq \mathbb{C}P^\infty \xrightarrow{\mathscr{A}(r)} G/O_{(2)},$$

where $\mathscr{A}(r)$ gives the standard solution to the $r$-local Adams conjecture for bundles of the form $\psi^r\xi - \xi$. The map $H^2(\mathbb{C}P^\infty; \mathbb{Z}_2) \to H^2(L(q;\ ra); \mathbb{Z}_2)$ is an isomorphism, so everything reduces to computing the $k_2$ class for $\mathscr{A}(r)$. Formulas of Brumfiel and Madsen [7, Proposition 6.1, p. 154; see also (8.6), p. 163] show that this class is zero if $r \equiv \pm 1 \bmod 8$ and nonzero if $r \equiv \pm 3 \bmod 8$. □

Putting all of this together, we obtain a complete formula for the normal invariants of Sieradski's simple homotopy equivalences.

**Theorem 3.8.** *Let $q > 1$ be given, let $t$ and $a$ be prime to $q$ so that $ta \equiv 1 \bmod q^6$, and let*

$$G: L(q;\ a) \times L(q;\ a) \to L(q;\ a^2) \times L(q;\ 1)$$

*be the homotopy equivalence in* [31]. *Then the normal invariant* $\eta(G)$ *is given as follows*:

(i) *If q is odd, it is trivial.*

(ii) *If q is even, it is trivial if* $a \equiv \pm 1 \bmod 8$ *and nontrivial if* $a \equiv \pm 3 \bmod 8$. *In the latter case, the normal invariant is given by the sum of the images of the generators of* $H^2(L(q;\ a^2);\mathbb{Z}_2)$ *and* $H^2(L(q;\ 1);\mathbb{Z}_2)$ *under the homomorphisms induced by the projections*:

$$L(q;\ a^2) \times L(q;\ 1) \to L(q;\ a^2),$$

$$L(q;\ a^2) \times L(q;\ 1) \to L(q,\ 1),$$

*respectively.*

**Proof.** By Proposition 3.5 it suffices to analyze the normal invariants on the four maps given by $\psi^a \times \psi^t$ and (3.2)–(3.4), and by Proposition 3.7, it suffices to show that the last three of these make no contribution to the normal invariant. Let $L = L(q,\ a^2)$ and $L' = L(a,\ 1)$. Consider first the map in (3.4) from $S^3 \times S^3$ to $L \times L'$, whose refined normal invariant construction can be determined from the factors. Once again we have maps $S^3 \to L$ so it suffices to localize at 2 and assume $q$ is even. Since $S^3 \to L$ is a multiple of the universal covering map and the latter is $S$-dual to the transfer $L \to S^0$ (cf. [3]), it follows that we are considering the $S$-dual to a map of the form

$$S^3 \times S^3 \xrightarrow{k \times 1} S^3 \times S^3 \xrightarrow{\pi \times \pi'} L \times L'.$$

Using the naturality of the Hopf construction

$$\mathrm{Maps}(X \times Y, Z) \to \mathrm{Maps}(X * Y, SZ),$$

we see that the normal invariant is merely $k$ times the smash product of the transfer from $L$ and $L'$ to $S^0$. But $ta \equiv 1 \bmod q^6$ implies $ta = kq^2 + 1$, where $q^4$ divides $k$, and each transfer has exponent dividing $q^2$. Therefore, this component of the normal invariant is trivial, which means that $S^3 \times S^3 \to L \times L'$ extends to a map of some parallelizable compact 7-manifold $U^7 \to L \times L'$.

To complete the proof it suffices to show that one can reframe the domains of the maps in (3.3) and (3.4) so that the associated elements in the framed bordism group $\Omega_6^{fr}(L \times L')$ are zero. The arguments in both cases are similar, so we shall focus on (3.3) which is slightly more subtle ($\mu'$ is less explicit than $\mu$). If $L'' = L(q, a)$ then this normal invariant piece is $S$-dual to an $S$-map

$$S^6 \xrightarrow{\deg 1} S^3 \times L'' \xrightarrow{1 \times \psi} S^3 \times L \xrightarrow{\mu'} L \subseteq L \times L'$$

and since the inclusion $L \subseteq L \times L'$ is a retract, it suffices to show that the $S$-dual of the composite $S^6 \to L$, which corresponds to an $S$-map $\theta$ from $S^3 \vee (S^4 \cup_q e^5)$ to $S^0$, is homotopically trivial or at least negligible. The latter will hold if we can show that $\theta$ lies in the image of the $J$-homomorphism from $[S^3 \vee (S^4 \cup_q e^5), SO]$ to $\{S^3 \vee (S^4 \cup_q e^5), S^0\}$. Since $\pi_4 = \pi_5 = \pi_4(SO) = \pi_5(SO) = 0$, this reduces to the surjectivity of the classical $J$-homomorphism from $\pi_3(SO)$ to $\pi_3$, which is well known. This completes the proof of Theorem 3.7. $\square$

## 4. More diffeomorphic squares of lens spaces

In this section, we shall prove that there are at most two homeomorphism/diffeomorphism classes of spaces of the form $L(q;\ a) \times L(q;\ a)$ if $q$ is divisible by 8 and exactly one class otherwise. The first step is to prove that $L(q;\ a) \times L(q;\ a)$ is diffeomorphic to $L(q;\ b) \times L(q;\ b)$ if $L(q;\ a)$ is homotopy equivalent to $L(q;\ b)$.

**Proposition 4.1.** *If $M$ and $N$ are homotopy equivalent of three-dimensional lens spaces. Then $M \times M$ and $N \times M$ are diffeomorphic.*

**Proof.** If $L(q;\ a)$ and $L(q;\ b)$ are homotopy equivalent, then by the homotopy classification of lens spaces we know that $\pm ab$ is a square mod $q$. Since elementary considerations imply that $a$ and $b$ are congruent mod $q$ to integers $a$ and $b$ such that $\pm ab$ is a perfect square mod $8q$, without loss of generality, we may assume that $\pm ab$ is a perfect square mod $8q$. Thus there is a homotopy equivalence $\bar{f}{:}L(8q;\ a) \to L(8q;\ b)$. Let $f{:}L(q;\ a) \to L(q;\ b)$ be the canonical lifting of $\bar{f}$.

Write $N = L(q;\ a)$ and $M = L(q;\ b)$. The Whitehead torsion $\tau(f)$ of $f$ is generally nontrivial, but the Whitehead torsion of $f \times id{:}N \times M \to M \times M$ is trivial because of the product formula (cf. [8, p. 77]):

$$\tau(f \times id) = \chi(N) \cdot i_* \tau(f) + \chi(N) \cdot j_* \tau(id).$$

Here $i,j{:}N \to N \times N$ are the inclusions onto first and second factor in $N \times N$, and $\chi(N) = 0$ is the Euler characteristic of $N$.

By Theorem 2.1 the simple homotopy equivalence $f \times id_M$ is homotopic to a diffeomorphism if its normal invariant vanishes. Therefore, we have to show that $f \times id{:}N \times M \to M \times M$ has trivial normal invariant; i.e.,

$$\eta([N \times M, f \times id]) = 0 \in [M \times M, G/O].$$

To see this, consider the "long exact surgery sequence in dimension 3" (cf. [10, p. 200]):

$$\Sigma_{\mathrm{Top}}(M) \xrightarrow{\eta_3} [M;G/\mathrm{Top}] \xrightarrow{\theta_3} L_3^h(\mathbb{Z}_{2^k}).$$

Here the set of $\Sigma_{\mathrm{Top}}(N)$ is defined in terms of homology $\mathbb{Z}[\mathbb{Z}_{2^k}]$ equivalences and corresponding $H$-cobordism classes (i.e., [10, p. 200]). Note that $[M;G/O] \approx [M;G/\mathrm{Top}] \approx H^2(M,\mathbb{Z}_2)$ because $M$ is three-dimensional. Therefore,

$$[M;G/O] \approx [M;G/\mathrm{Top}] \approx H^2(M;\mathbb{Z}_2) \approx \mathbb{Z}_2$$

and this copy of $\mathbb{Z}_2$ injects into $L_3^h(\mathbb{Z}_{2^k}) \approx \mathbb{Z}_2$ (once again Theorem A in [12]). Since $f: N \to M$ is a homotopy equivalence, it follows that $\theta_3([f]) = 0$, which implies $\eta_3([f]) = 0$. Since $f$ is normally cobordant to the identity, so is $f \times id$. This finishes the proof of Proposition 4.1. □

**Corollary 4.2.** *If $M$ and $N$ are homotopy equivalent lens spaces then $M \times M$ and $N \times N$ are diffeomorphic. Furthermore, if $k_M$ and $k_N$ are canonical polarizations of $M$ and $N$ one can choose the* *diffeomorphism $F$ so that the diagram below is homotopy commutative for suitable homotopy self-equivalences $g, g'$ of $K(\mathbb{Z}_q,1)$*:

$$\begin{array}{ccc} N \times N & \xrightarrow{\quad F \quad} & M \times M \\ \Big\downarrow{\scriptstyle k_N \times k_N} & & \Big\downarrow{\scriptstyle k_M \times k_M} \\ K(\mathbb{Z}_q,1) \times K(\mathbb{Z}_q,1) & \xrightarrow{g \times g'} & K(\mathbb{Z}_q,1) \times K(\mathbb{Z}_q,1) \end{array}$$

**Proof.** As in the proof of Proposition 4.1 write $N = L(q;\ a)$ and $M = L(q;\ b)$, and likewise take $f$ to be the lifting of a homotopy equivalence $\bar{f}$:$L(8q,\ a) \to L(8q,\ b)$. Let $\bar{g}$ be a homotopy inverse to $\bar{f}$, and let $g$:$M \to N$ be the canonical lifting of $\bar{g}$. Then Proposition 4.1 implies that both $f \times id_M$ and $id_N \times g$ are homotopic to diffeomorphisms, and therefore the composite

$$F_0 = (id_N \times g)\circ(f \times id_M) : N \times N \to M \times M$$

is also homotopic to a diffeomorphism, say $F$. By construction $F_0$ fits into a commutative diagram of the type described above, and therefore $F$ fits into a homotopy commutative diagram of the same type. □

We shall combine this with Theorem 3.8 to prove the main result of this section.

**Theorem 4.3.** (i) *If $q$ is not divisible by* 8, *then for each $a, b$ prime to $q$, the lens space squares $L(q;\ a) \times L(q;\ a)$ and $L(q;\ b) \times L(q;\ b)$ are diffeomorphic.*

(ii) *If $q$ is divisible by* 8 *and $a, b$ are prime to $q$ such that* $a \equiv \pm b \bmod 8$, *then the lens space squares $L(q;\ a) \times L(q;\ a)$ and $L(q;\ b) \times L(q;\ b)$ are diffeomorphic.*

In particular, the conclusion of (ii) implies that every lens space square under consideration is diffeomorphic to

$$L(q;1) \times L(q;1) \quad \text{or} \quad L(q;c) \times L(q;c),$$

where $c$ is prime to $q$ and satisfies $c \equiv 3 \bmod 8$.

**Proof.** There are three cases:

*Case 1*: Suppose $q$ is divisible by 8 and $a$ and $b$ are congruent to $\pm 1 \bmod 8$. The results of Section 3 yield a simple homotopy equivalence $\bar{h}_0$:$L(2q;a) \times L(2q;a) \to L(2q;a^2) \times L(2q;1)$ with trivial normal invariants, and if we compose this with the diffeomorphism $L(2q;a^2) \times L(2q;1) \to L(2q;1) \times L(2q;1)$ from Proposition 4.1, we obtain another simple homotopy equivalence $\bar{h}_0$:$L(2q;a) \times L(2q;a) \to L(2q;1) \times L(2q;1)$ with trivial normal invariant. If $h$ is the lifting of $\bar{h}$ to a simple homotopy equivalence $L(q;a) \times L(q;a) \to L(q;1) \times L(q;1)$, then by Theorem 2.1 $h$ is homotopic to a diffeomorphism, say $H$. Similarly, there is a diffeomorphism $K$ from $L(q;b) \times L(q;b)$ to $L(q;1) \times L(q;1)$, and therefore $K^{-1}H$ is a diffeomorphism from $L(q;a) \times L(q;a) \to L(q;b) \times L(q;b)$.

*Case 2*: Suppose $q$ is not divisible by 8. By the results of Section 3 we may as well assume that $q$ is even and hence $4q$ is divisible by 8. Then $a$ and $b$ are odd, and we can choose integers $a'$ and $b'$ so that $a \equiv a' \bmod q$, $b \equiv b' \bmod q$, $a \equiv \pm 1 \bmod 8$ and $b \equiv \pm 1 \bmod 8$. By Case 1 we know that

$$L(4q;a') \times L(4q;a') \quad \text{and} \quad L(4q;b') \times L(4q;b')$$

are diffeomorphic. Since their respective covering spaces associated to the subgroup

$$\mathbb{Z}_q \times \mathbb{Z}_q \subseteq \mathbb{Z}_{4q} \times \mathbb{Z}_{4q}$$

are $L(q;a) \times L(q;a)$ and $L(q;b) \times L(q;b)$, respectively, we may lift the diffeomorphism given by Case 1 to obtain a diffeomorphism between the latter two manifolds.

*Case 3*: Suppose that $q$ is divisible by 8 and $a$ and $b$ are congruent to $\pm 3 \bmod 8$. In this case, the argument in Case 1 yields simple homotopy equivalences

$$\bar{h}\colon L(2q;a) \times L(2q;a) \to L(2q;1) \times L(2q;1),$$

$$\bar{k}\colon L(2q;b) \times L(2q;b) \to L(2q;1) \times L(2q;1)$$

with the same nonzero normal invariant; specifically, $\eta(\bar{h}) = \eta(\bar{k})$ is 2-primary and detected by its $k_2$ Sullivan class, which is the sum of the generators for

$$\mathbb{Z}_2 \approx H^0(L(q;1),\mathbb{Z}_2) \otimes H^2(L(q;1);\mathbb{Z}_2)$$

and

$$\mathbb{Z}_2 \approx H^2(L(q;1),\mathbb{Z}_2) \otimes H^2(L(q;1);\mathbb{Z}_2)$$

in the Künneth Formula decomposition of $H^2(L(q;1) \times L(q;1);\mathbb{Z}_2)$. Let $\bar{\ell}$ be a homotopy inverse to $\bar{k}$. Then a couple of applications of the normal invariant formulas

$$\eta(g \circ f) = \eta(f) + f^{*-1}\eta(g)$$

implies that $\eta(\bar{\ell} \circ \bar{h}) = 0$, and therefore by Theorem 2.1 the lifted homotopy equivalence $\ell \circ h\colon L(q;a) \times L(q;a) \to L(q;b) \times L(q;b)$ is homotopic to a diffeomorphism. $\square$

## 5. Completion of the proof

As noted after the statement of Theorem 4.3, there are at most two diffeomorphism classes of squares of lens spaces with fundamental group $\mathbb{Z}_q \oplus \mathbb{Z}_q$ if $q$ is divisible by 8; namely, every such square is diffeomorphic to either $L(q;1) \times L(q;1)$ or $L(q;c) \times L(q;c)$ where $c$ is prime to $q$ and $c \equiv 3 \bmod 8$. By the results of Sections 1 and 3, there is a simple homotopy equivalence $h\colon L(q;c) \times L(q;c) \to L(q;1) \times L(q;1)$ whose normal invariant $\eta(h)$ satisfies the following two conditions:

(5.1) "The Sullivan $k_2$ class of $\eta(h)$ in $H^2(L(q;1) \times L(q;1);\mathbb{Z}_2)$ is the sum of the generators of $H^2(L(q;1);\mathbb{Z}_2) \otimes \mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes H^2(L(q;1);\mathbb{Z}_2)$ with respect to the Künneth Formula splitting of $H^2(L(q;1) \times L(q;1);\mathbb{Z}_2)$".

(5.2) "The localization of $\eta(h)$ away from 2 is trivial, and the higher 2-local Sullivan classes of $\eta(h)$ are also trivial".

In order to complete the proof of the main result, we need to modify $h$ to obtain a new simple homotopy equivalence that is homotopic to a diffeomorphism. Such a map is homotopic to a composite $g \circ h$ where the factor $g$ is a simple homotopy self-equivalence of $L(q;1) \times L(q;1)$ and the formula for normal invariants of a composite implies that $g^*\eta(g) = \eta(h)$ if $g \circ h$ is homotopic to a diffeomorphism. Since Theorem 2.1 implies that $g \circ h$ is homotopic to a diffeomorphism if and only if $\eta(g \circ h) = 0$, we obtain the following reduction of the proof of the main result:

**Proposition 5.3.** *Let g be a simple homotopy self-equivalence of* $L(q;1) \times L(q;1)$ *such that g induces the identity on homology and cohomology, and suppose that its normal invariant* $\eta(g)$ *satisfies* (5.1) *and* (5.2). *Then* $g \circ h$ *is homotopic to a diffeomorphism.*

This follows because the homological condition implies that $g^*\eta(g) = \eta(g)$ and the hypothesis implies $\eta(g) = \eta(h)$, so that Theorem 2.1 and the composition formula imply $g \circ h$ is homotopic to a diffeomorphism.

The main result thus reduces to proving the following statement:

**Proposition 5.4.** *If q is even, then there is a simple homotopy self-equivalence g of* $L(q;1) \times L(q;1)$ *such that* $\eta(g)$ *satisfies* (5.1) *and* (5.2).

**Proof.** Let $f$ be the homotopy self-equivalence of $S^2 \times S^2$ given by the following composite:

$$S^2 \times S^2 \xrightarrow{pinch} S^2 \times S^2 \vee S^4 \xrightarrow{1 \vee F} (S^2 \times S^2) \vee (S^2 \times S^2) \xrightarrow{fold} S^2 \times S^2.$$

Here $F: S^4 \to S^2 \times S^2$ is a map whose projection on both factors represent generators of $\pi_4(S^2) \approx \mathbb{Z}_2$. Standard formulas for normal invariants (going back to Novikov's paper [24, pp. 327–328], especially Lemma 7.6 and Remark 1, p. 328) imply that the normal invariant

$$\eta(f) \in [S^2 \times S^2, G/O] \approx \pi_2(G/O) \oplus \pi_2(G/O) \oplus \pi_4(G/O)$$

is the sum of the generators of the first two summands. By construction, $f | S^2 \vee S^2$ is the identity.

Consider the principal $T^2$ bundle $\xi$ over $S^2 \times S^2$ given by the square of the principal circle bundle $S^1 \to L(q;\, 1) \xrightarrow{\alpha} S^2$. Since these torus bundles are classified by two-dimensional integral cohomology classes, and $f$ is the identity on $S^2 \vee S^2$, it follows that $f^*\xi$ is isomorphic to $\xi$. This in turn implies that $f$ lifts to an $S^1$-equivariant simple homotopy self-equivalence $g$ of $L(q;\, 1) \times L(q;\, 1)$ whose restriction to $L(q;1) \vee L(q;1)$ is the identity. For such principal bundle maps one has a normal invariant formula

$$\eta(g) = (\alpha \times \alpha)^* \eta(f)$$

and since $\alpha^*: H^2(S^2;\mathbb{Z}_2) \to H^2(L(q;1);\mathbb{Z}_2)$ is an isomorphism, it follows that $\eta(g) = (\alpha \times \alpha)^* \eta(f)$ satisfies (3.1) and (3.2). This proves the proposition and also completes the proof of the main result. □

## 6. Some further results

*Higher powers of lens spaces*: Clearly, there is an analog of Ulam's question for $n$th powers of spaces for every integer $n \geqslant 3$ (and of course one can even replace positive integers by arbitrary cardinal numbers). As indicated in Theorem B, our methods show that the classification of nontrivial lens space powers up to homeomorphism or diffeomorphism is exactly the same as the classification up to homotopy equivalence due to Sieradski [31]; namely, there is exactly one homotopy type of $n$th powers if $n$ is even, and there is a 1–1 correspondence between diffeomorphism (resp., homeomorphism) types of $n$th powers and ordinary homotopy types of lens spaces if $n \geqslant 3$ is odd.

**Proof of Theorem B.** If $n$ is even, write $n = 2m$. By the results of the previous sections we know that $L \times L$ is diffeomorphic to $L' \times L'$ if $L$ and $L'$ have isomorphic fundamental groups. The $m$th Cartesian power of such a diffeomorphism then yields a diffeomorphism from $\prod^{2m} L$ to $\prod^{2m} L'$. Conversely, if the two products are diffeomorphic then their fundamental groups are isomorphic, and by the classification of finite abelian groups the fundamental groups of $L$ and $L'$ must also be diffeomorphic.

Suppose now that $n \geqslant 3$ is odd. Since diffeomorphic manifolds are homotopy equivalent, the existence of a homeomorphism from $\prod^n L$ to $\prod^n L'$ implies that $L$ and $L'$ are homotopy equivalent by Theorem 3 and the Corollary [31, p. 22]. To prove that $\prod^n L$ is diffeomorphic to $\prod^n L'$ if $L$ and $L'$ are homotopy equivalent, we use the diffeomorphism $L \times L' \approx L' \times L'$ given by Proposition 4.1. Specifically, for each $k$ such that $1 \leqslant k \leqslant n$, take the product of the diffeomorphism $L \times L' \approx L' \times L'$ given by 4.1 with $k-1$ copies of the identity on $L$ and $n-k-1$ copies of the identity on $L'$. This yields a diffeomorphism

$$\Phi_k : \Big(\prod^{k-1} L \times \prod^{n-k-1} L'\Big) \to \Big(\prod^{k} L \times \prod^{n-k} L'\Big),$$

where $\prod^0 A$ is defined to be a point for the sake of notational uniformity. The composite of these diffeomorphisms then yields a diffeomorphism from $\prod^n L$ to $\prod^n L'$. $\square$

*Squares of higher-dimensional manifolds*: Finally, for each dimension $\geqslant 4$ here is a construction of two closed $n$-manifolds $M^n$, $N^n$ that are not homotopy equivalent but have diffeomorphic squares. This is based upon the following elementary observation.

**Proposition 6.1.** *Let $L$ and $L'$ be three-dimensional lens spaces and let $k \neq 3$ be a positive integer. Then $L \times S^k$ and $L' \times S^k$ are homotopy equivalent only if $L$ and $L'$ are. Furthermore, the same is true for: $L \times S^2 \times S^1$.*

**Proof.** A homotopy equivalence induces an isomorphism of fundamental groups, and the torsion subgroups of the (abelian) fundamental groups of $L \times S^k$ and $L' \times S^k$ (resp. $L \times S^2 \times S^1$ and $L' \times S^2 \times S^1$) are equal to the fundamental groups of $L$ and $L'$, respectively. Therefore, the fundamental groups of $L$ and $L'$ are isomorphic, if $L \times S^k$ and $L' \times S^k$ are homotopy equivalent and likewise if $L \times S^2 \times S^1$ and $L' \times S^2 \times S^1$ are homotopy equivalent. In fact, it follows that the appropriate composite

$$\varphi : L \subset L \times S^k \simeq L' \times S^k \to L',$$

$$\varphi : L \subset L \times S^2 \times S^1 \simeq L' \times S^2 \times S^1 \to L',$$

induces an isomorphism of fundamental groups.

Suppose now that $L \times S^2 \times S^1$ and $L' \times S^2 \times S^1$ are homotopy equivalent. It follows that the infinite cyclic coverings corresponding to projections onto $S^1$ are also homotopy equivalent; since these are homeomorphic to $L \times S^2 \times \mathbb{R}$ and $L' \times S^2 \times \mathbb{R}$, respectively, it follows that $L \times S^2$ and $L' \times S^2$ are homotopy equivalent in this case. Thus it suffices to prove the assertion about products with $S^k$ for $k \neq 3$.

We now have a situation where $L \times S^k$ and $L' \times S^k$ are homotopy equivalent for some positive integer $k \neq 3$; let $h$ be such a homotopy equivalence. Consider now the map of universal coverings from $S^3 \times S^k$ to itself induced by $h$. This map is also a homotopy equivalence, and as such it induces an isomorphism on three-dimensional integral cohomology. Since $k \neq 3$, it follows that the latter implies that the lifting of $\varphi$ to the universal coverings induces an isomorphism in cohomology from $S^3$ to itself, and it follows that $\varphi$ must be a homotopy equivalence. □

**Corollary 6.2.** *If L and L′ are three-dimensional lens spaces that have isomorphic fundamental groups but are not homotopy equivalent, and n is a positive integer* $\neq 3$, *then* $L \times S^n$ *and* $L' \times S^n$ *are nonhomeomorphic manifolds whose squares are diffeomorphic. Furthermore,* $L \times S^2 \times S^1$ *and* $L' \times S^2 \times S^1$ *are homotopy inequivalent manifolds that have diffeomorphic squares.*

This follows by combining Proposition 6.1 and Theorem A. □

*Remarks on third powers in higher dimensions*: The preceding does not provide any examples of nonhomeomorphic manifolds with diffeomorphic third powers. In fact, such manifolds exist in every dimension; for example, if $L$ and $L'$ are nonhomeomorphic lens spaces then the products $L \times T^k$ and $L' \times T^k$ with the $k$-torus are not homeomorphic. The proof involves some lengthy additional digressions, so the argument will not be given here.

## Acknowledgements

Stimulating and inspiring discussions with Dr. Witold Rosicki were very influential on the research presented in this paper.

## References

[1] A. Bak, Odd dimension surgery groups of odd torsion groups vanish, Topology 14 (1975) 367–374.
[2] A. Bak, The computation of surgery groups of finite groups with abelian 2-hyperelementary subgroups, in: M. R. Stein (Ed.), Algebraic $K$-Theory, Proceedings of the Conference at Northwestern University, 1976, Lecture Notes in Mathematics, Vol. 551, Springer, Berlin, 1976, pp. 384–409.
[3] J.C. Becker, D.H. Gottlieb, The transfer map and fiber bundles, Topology 14 (1975) 1–12.
[4] J.M. Boardman, R.M. Vogt, Homotopy-everything $H$-spaces, Bull. Amer. Math. Soc. 74 (1968) 1117–1122.
[5] K. Borsuk, Sur la décomposition des polyédres en produits cartésiens, Fund. Math. 31 (1938) 137–148.
[6] W. Browder, Surgery on simply-connected manifolds, Ergeb. Math. (2) No. 65, Springer, Berlin, 1972.

[7] G. Brumfiel, I. Madsen, Evaluation of the transfer and the universal surgery classes, Invent. Math. 32 (1976) 133–169.
[8] M. Cohen, A Course in Simple Homotopy Theory, Graduate Texts in Mathematics, Vol. 10, Springer, Berlin, 1973.
[9] R.M. Fox, On a problem of S. Ulam concerning Cartesian products, Fund. Math. 34 (1947) 278–287.
[10] M. Freedman, F. Quinn, Topology of 4-Manifolds, Princeton Math. Series, Vol. 39, Princeton University Press, Princeton, 1990.
[11] J. Glimm, Two Cartesian products which are Euclidean spaces, Bull. Soc. Math. France 88 (1960) 131–135.
[12] T. Hambleton, R.J. Milgram, L. Taylor, B. Williams, Surgery with finite fundamental group, Proc. London Math. Soc. 56 (3) (1988) 349–379.
[16] S. Kwasik, R. Schultz, Inductive detection of homotopy equivalences of manifolds, $K$-Theory 11 (1997) 287–306.
[17] S. Kwasik, R. Schultz, Cartesian powers of 3-manifolds, preprint, Tulane University and UC Riverside, 1999.
[18] K.W. Kwun, Products of Euclidean spaces modulo an arc, Ann. Math. 79 (1964) 104–108.
[19] I. Madsen, L. Taylor, B. Williams, Tangential homotopy equivalences, Comment. Math. Helv. 55 (1980) 445–484.
[20] D.R. McMillan Jr., Cartesian products of contractible open manifolds, Bull. Amer. Math. Soc. 67 (1961) 510–514.
[21] W. Metzler, Diffeomorphismen zwischen Produkten mit dreidimensionalen Linsen-räumen als Faktoren, Dissertationes Math. Rozprawy 65 (1969) 1–60.
[22] W. Metzler, Diffeomorphismen zwischen Produkten aus Quotientmannigfaltigkeiten von $S^3$, J. Reine Angew. Math. 248 (1971) 186–192.
[23] Y.V. Muranov, Obstructions to surgery of double coverings, Math. USSR, Sb. 59 (2) (1988) 339–348.
[24] S.P. Novikov, Homotopically equivalent smooth manifolds I, Izv. Akad. Nauk. SSST Ser. Mat. 28 (1964) 365–474 (in Russian) (English transl., AMS Translations 48 (1965) 271–396).
[25] R. Oliver, Whitehead Groups of Finite Groups, London Math. Soc. Lecture Notes, Vol. 132, Cambridge University Press, Cambridge, UK, 1989.
[26] W. Rosicki, On a problem of S. Ulam concerning Cartesian squares of 2-dimensional polyhedra, Fund. Math. 127 (1986) 101–125.
[27] C.P. Rourke, The Hauptvermutung according to Casson and Sullivan, in: A.A. Ranicke, A.J. Casson, D.P. Sullman, M.-A. Armstrong, C.P. Rourke, G.E. Cooke (Eds.), The Haupt-vermutung Book, K-Monographs in Mathematics, Vol. 1, Kluwer Academic Publishing, Dordrecht, 1996, pp. 129–164.
[28] R. Schultz, Differentiable group actions on homotopy spheres: I Differential structure and the knot invariant, Invent. Math. 31 (1975) 105–128.
[29] R. Schultz, Finding framed $\mathbb{Z}_p$-actions on exotic spheres, in: J. Dupont, I. Madsen (Eds.), Algebraic Topology, Aarhus 1978 (Proceedings), Lecture Notes in Mathematics, Vol. 763, Springer, Berlin, 1979, pp. 591–603.
[30] L. Siebenmann, Topological manifolds, in Actes, Congr. Int. Math. (Nice, 1970), T. 2, Gauthier-Villars, Paris, 1971, pp. 143–163.
[31] A. Sieradski, Non-cancellation and related phenomena for lens spaces, Topology 17 (1978) 85–95.
[34] S. Ulam, Problem 56, Fund. Math. 20 (1933) 285.
[36] C.T.C. Wall, On the classification of hermitian forms: VI. Group rings, Ann. Math. 103 (1976) 1–80.


## Further reading


[13] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Annals of Mathematics Studies, Vol. 88, Princeton University Press, Princeton, 1977.
[14] S. Kwasik, R. Schultz, On $s$-cobordisms of metacyclic prism manifolds, Invent. Math. 97 (1989) 523–552.
[15] S. Kwasik, R. Schultz, Vanishing of Whitehead torsion in dimension four, Topology 31 (1992) 735–756.
[32] L. Taylor, B. Williams, Surgery on closed manifolds, preprint, University of Notre Dame, 1980.
[33] W. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry Bull. Amer. Math. Soc. (2)6 (1982) 357–381.
[35] C.T.C. Wall, Surgery on Compact Manifolds, London Math. Soc. Monographs, No. 1, Academic Press, New York, 1970.